Математический анализ 1 — МИЭФ, 2020 midterm

МИЭФ | Математический анализ 1 | 2020 | midterm

Рисунки пока рендерятся в тестовом режиме и могут отличаться от исходных материалов.

QUESTION 1

On the grid, sketch the graph of a function $y = f(x)$ satisfying all the conditions below. Use a solid dot ● or an open dot ○ where necessary to indicate whether a point belongs to the graph.

(a) f is defined for every x and $f(-3) = 0$.

(b) $\lim_{x \to -\infty} f(x) = 0$.

(c) The graph has an asymptote $y = 1 + x$.

(d) $\lim_{x \to 3+} f(x) = 5$.

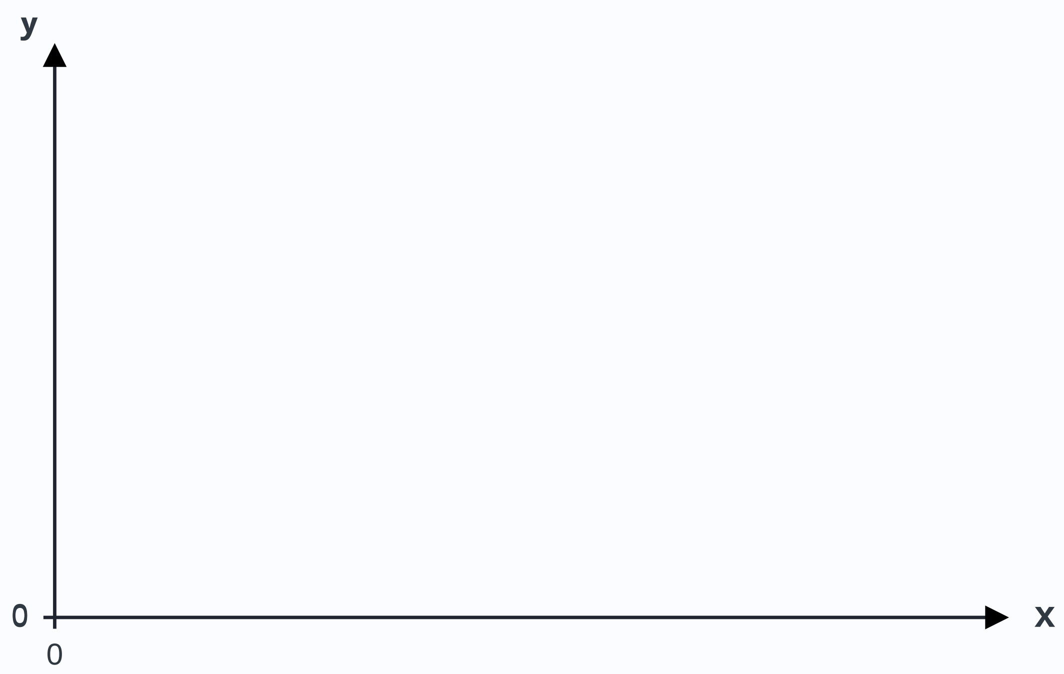
(e) $\lim_{x \to 3} f(x)$ exists, but f is not continuous at $x = 3$.

(f) $\lim_{x \to 0+} f(x)$ exists.

(g) $\lim_{x \to 0-} f(x) = +\infty$.

(h) f has exactly three points of discontinuity: one removable, one jump, and one other.

(i) The Intermediate Value Theorem does not apply to $f(x)$ on $[-7, -3]$.



Grid for constructing f

QUESTION 2

Consider

$$f(x) = \frac{\ln(x^2)}{x^2 - 1} - x\ln(2x+3).$$

(a) Find the domain D of f.

(b) There are some points p such that $\lim_{x\to p} f(x)$ exists, but $p \notin D$. Find all such points and the corresponding limits.

(c) Find all vertical and slant asymptotes of the graph $y = f(x)$. Justify.

(d) Explain why there is at least one solution of $f(x) = 0$ on $(0, 1)$.

QUESTION 3

Consider an infinite sequence $\{a_n\}$ where

$$a_n = \frac{6 - 5n}{10n - 8}.$$

(a) Find $L = \lim_{n\to+\infty} a_n$.

(b) Find the smallest index N such that all terms a_n with $n > N$ lie inside $(L - 0.01, L + 0.01)$.

(c) The sequence $\{b_n\}$ is defined recursively by $b_1 = -24$, $b_{n+1} = a_n b_n$. Compute b_2 and b_3. Is $\{b_n\}$ monotonic?

(d) Is $\{b_n\}$ convergent? If yes, find $\lim_{n\to+\infty} b_n$; otherwise explain why.

QUESTION 4

At time $t = 0$, robots Ace and Bdf start moving from the origin O. Every minute each robot makes exactly one step of size 1 either up or right. Their algorithms are:

$$\text{Ace: } 1 \rightarrow,\ 1 \uparrow,\ 1 \rightarrow,\ 1 \uparrow,\ 2 \rightarrow,\ 2 \uparrow,\ 4 \rightarrow,\ 4 \uparrow, \ldots$$

$$\text{Bdf: } 2 \uparrow,\ 1 \rightarrow,\ 4 \uparrow,\ 3 \rightarrow,\ 6 \uparrow,\ 5 \rightarrow, \ldots$$

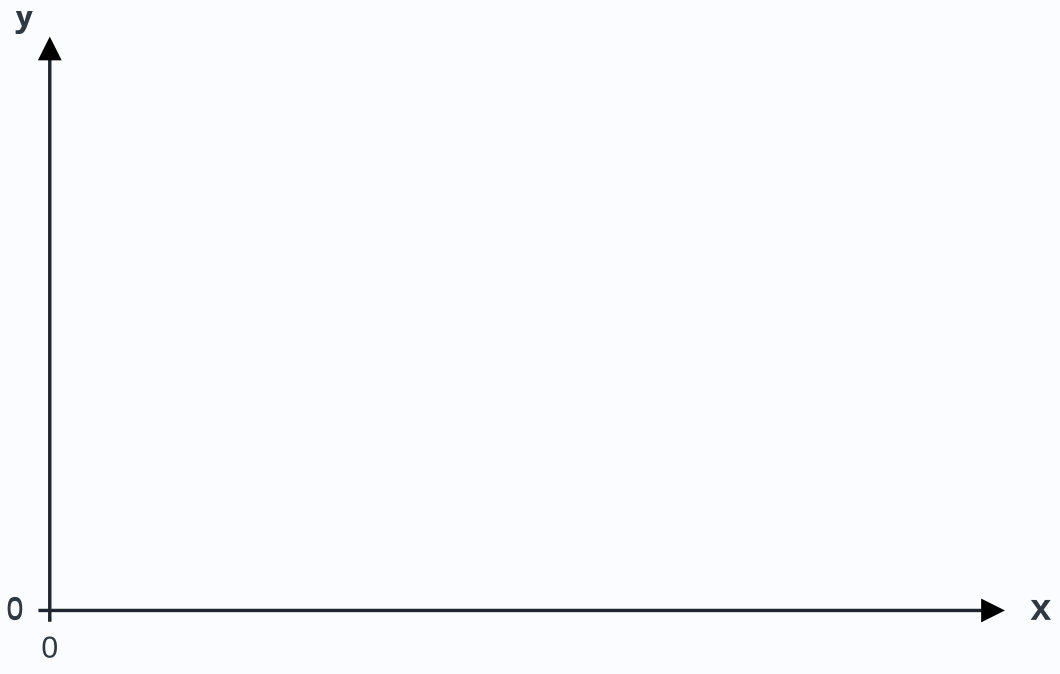
Here $k \uparrow$ means k steps up and $k \rightarrow$ means k steps right. Let A_n and B_n be their positions after n minutes, and let α_n and β_n be the slopes of OA_n and OB_n.

(a) Sketch the paths of the robots.

(b) Show that $\alpha_n \leq \beta_n$ for all $n > 0$.

(c) Is $\{\alpha_n\}$ convergent? Justify.

(d) Is $\{\beta_n\}$ convergent? Justify.



Robot paths for Ace and Bdf

QUESTION 5

Let $\{a_n\}$ and $\{b_n\}$ be two convergent infinite sequences. Which infinite sequences must be convergent?

I. $\{\max(a_n, b_n)\}$.

II. $\{a_n b_{n+2019} - a_{n+2019} b_n\}$.

III. $a_1, b_1, a_2, b_2, \ldots, a_n, b_n, a_{n+1}, b_{n+1}, \ldots$.

1. I, II and III.
2. I and II only.
3. I and III only.
4. II and III only.

QUESTION 6

Find k that makes f continuous at $x = -3$:

$$f(x) = \begin{cases} \dfrac{\sqrt{5x^2+5x-5} - \sqrt{4x^2+4x+1}}{x+3}, & x \neq -3, \\ k, & x = -3. \end{cases}$$

1. 0.5.
2. 0.1.
3. -0.5.
4. -0.1.

QUESTION 7

The set of all points (e^t, t), where t is real, is a graph of:

1. $y = e^{1/x}$.
2. $y = \ln x$.
3. $y = \frac{1}{e^x}$.
4. $y = \frac{1}{\ln x}$.

QUESTION 8

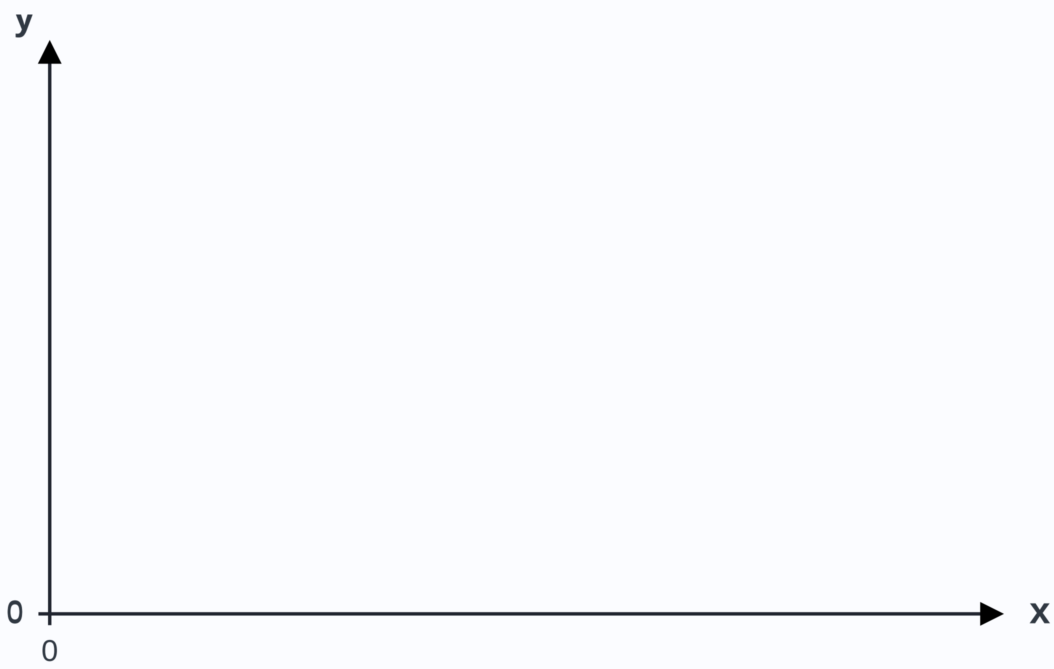
Find

$$\lim_{n \to \infty} \frac{(2n+1)(4n+1)(6-n)}{(5n+4)(4n+3)(3-2n)}.$$

1. $\frac{2}{15}$.
2. $\frac{1}{5}$.
3. $\frac{3}{20}$.
4. 0.

QUESTION 9

Which instruction, applied to the graph $y = \sin x$, produces the shown graph?



Transformed sine graph

Нужно проверить: "Exact phase is best checked against the source image."

1. Shift π units right, then compress horizontally by a factor of π.
2. Compress horizontally by a factor of π, then reflect in the x-axis.
3. Shift $\pi/2$ units right, then compress horizontally by a factor of π.
4. Compress horizontally by a factor of π, then shift $1/2$ unit left.

QUESTION 10

Which choice of δ is the largest that can be used successfully with arbitrary ε in an ε–δ proof of

$$\lim_{x \to 2}(1 - 3x) = -5?$$

1. $\delta = \frac{\varepsilon}{4}$.
2. $\delta = \frac{\varepsilon}{2}$.
3. $\delta = \frac{3\varepsilon}{2}$.
4. $\delta = \frac{\varepsilon}{5}$.

QUESTION 11

Suppose f is defined for all real numbers. Which condition ensures that f has an inverse function?

1. The graph $y = f(x)$ is symmetric with respect to the y-axis.
2. f is periodic.
3. f is continuous.
4. f is strictly decreasing.

QUESTION 12

Find a such that

$$\lim_{h\to 0} \frac{1}{h} \ln\left(\frac{a+h}{a}\right) = 2.$$

1. $\ln 2.$
2. $0.5.$
3. $2.$
4. $-\ln 2.$

QUESTION 13

Find $f \circ g \circ h$ if

$$f(x) = \frac{x}{x+1}, \qquad g(x) = e^x, \qquad h(x) = x^2.$$

1. $\frac{e^{2x}}{e^{2x}+1}.$
2. $e^{2x/(x+1)}.$
3. $\left(\frac{e^x}{e^x+1}\right)^2.$
4. $\frac{e^{x^2}}{e^{x^2}+1}.$

QUESTION 14

Find

$$\lim_{x \to 3-} \arcsin\left(\frac{|3x - x^2|}{x^2 - 9}\right).$$

1. $\frac{\pi}{6}$.
2. $\frac{\pi}{3}$.
3. $-\frac{\pi}{6}$.
4. $-\frac{\pi}{3}$.

QUESTION 15

Suppose f is odd and the graph $y = f(x)$ has asymptote $y = 26x - 10$ as $x \to +\infty$. Which is the asymptote as $x \to -\infty$?

1. $y = 26x - 10.$
2. $y = -26x - 10.$
3. $y = 26x + 10.$
4. $y = -26x + 10.$

QUESTION 16

Which sequences are convergent?

$$a_n = \frac{(2n-1)!}{(n+2019)!}, \qquad b_n = \frac{2^{n/2}}{n^{2020}}, \qquad c_n = \frac{n^4}{(n+1)!}.$$

1. $\{c_n\}$ only.
2. $\{a_n\}$ and $\{b_n\}$ only.
3. $\{b_n\}$ and $\{c_n\}$ only.
4. $\{a_n\}$ only.

QUESTION 17

Let g be continuous on $[0,1]$, with $g(0)=1$ and $g(1)=0$. Which statement is not necessarily true?

1. There exists $h \in [0,1]$ such that $g(h) = e^{\pi}$.
2. There exists $h \in [0,1]$ such that $g(h) = \frac{\pi}{e}$.
3. For all $h \in (0,1)$, $\lim_{x\to h} g(x) = g(h)$.
4. There exists $h \in [0,1]$ such that $g(h) \geq g(x)$ for all $x \in [0,1]$.

QUESTION 18

If $9^x - 8^x$ is equivalent to cx as $x \to 0$, then $c =$

1. $2\ln 3 - 3\ln 2$.
2. 1.
3. $3\ln 2 - 2\ln 3$.
4. 0.

QUESTION 19

A continuous positive function f has domain $x > 0$. If asymptotes are $x = 0$ and $y = 2$, which statement must be true?

1. $\lim_{x\to 0+} f(x) = 2$ and $\lim_{x\to+\infty} f(x) = 0$.
2. $\lim_{x\to 0+} f(x) = \infty$ and $\lim_{x\to+\infty} f(x) = 2$.
3. $\lim_{x\to 2} f(x) = \infty$ and $\lim_{x\to+\infty} f(x) = 2$.
4. $\lim_{x\to 0+} f(x) = \infty$ and $\lim_{x\to 2} f(x) = \infty$.

QUESTION 20

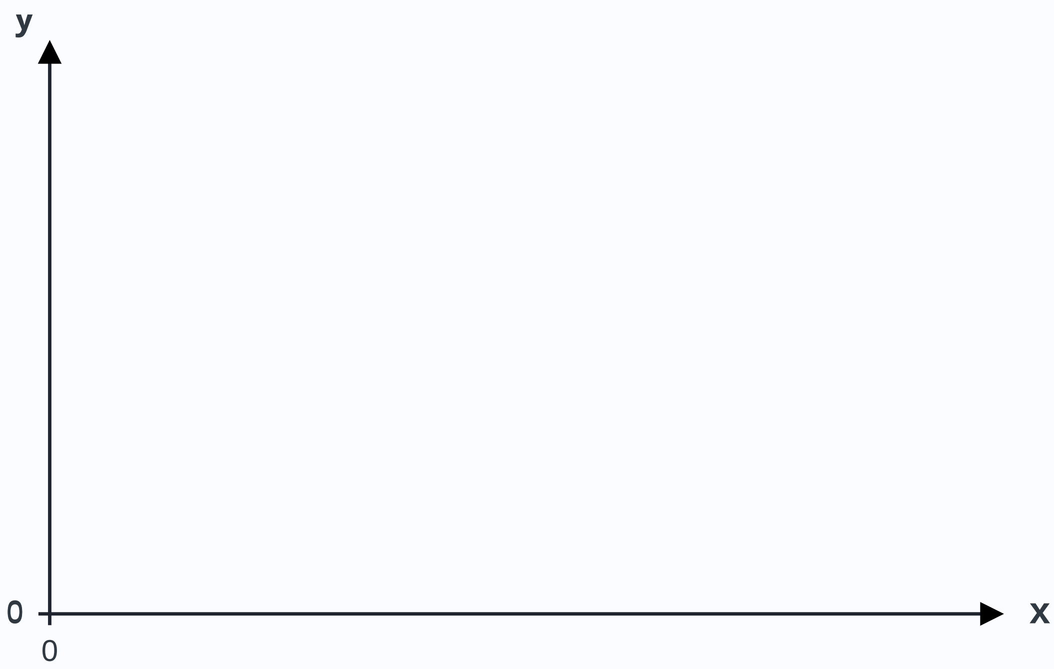
Compute

$$\lim_{x\to+\infty}\left(\frac{x+9}{x-11}\right)^{x+5}.$$

1. e^{10}.
2. e^{5}.
3. e^{20}.
4. 1.

QUESTION 21

Graphs of f and g are shown. Which statement is false?



Graphs of f and g used in composite product limits

Нужно проверить: "Exact coordinates should be checked visually if solving."

1. $\lim_{x \to 1} f(x)g(x+1)$ does not exist.
2. $\lim_{x \to 1} f(x) = 0$.
3. $\lim_{x \to 2} g(x)$ does not exist.
4. $\lim_{x \to 1} f(x+1)g(x)$ exists.

QUESTION 22

Which interval belongs to the domain of

$$f(x) = \tan(x-1)?$$

1. $\left(\frac{3\pi}{2}+1, \frac{5\pi}{2}+1\right)$.
2. $\left(\frac{5\pi}{2}-1, \frac{7\pi}{2}-1\right)$.
3. $\left(\frac{7\pi}{2}, \frac{9\pi}{2}\right)$.
4. $\left(-\frac{\pi}{2}-1, \frac{\pi}{2}-1\right)$.

QUESTION 23

The sequences x_n, y_n, z_n satisfy $x_n < y_n < z_n$. Which statements are false?

I. If x_n and z_n are convergent, then y_n is also convergent.

II. If $\lim z_n = \lim x_n = L \in \mathbb{R}$, then $\lim y_n = L$.

III. If all three sequences converge, then $\lim x_n < \lim y_n < \lim z_n$.

1. I and II only.
2. I and III only.
3. II only.
4. I only.

QUESTION 24

Which statement is true about the graph of

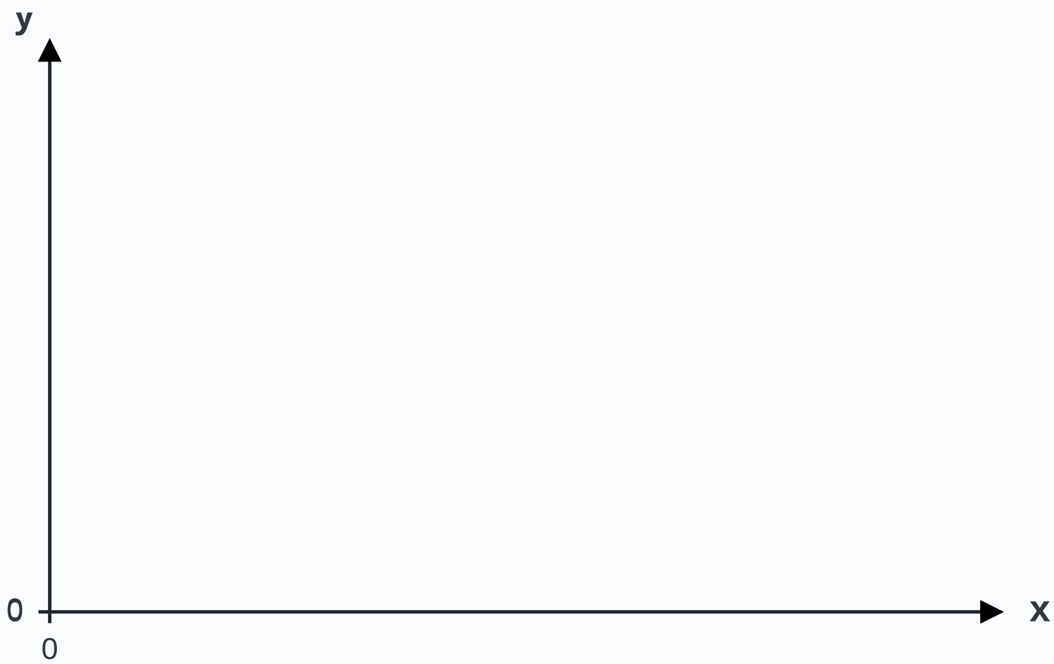
$$y = \frac{1}{\sqrt{1 - x^2}}$$

on $(-1, 1)$?

1. It is monotonic.
2. Its range is all positive real numbers.
3. It is continuous.
4. It is bounded.

QUESTION 25

Function f is defined for all real x. The figure shows part of $y = f(x)$, including the only discontinuity at $x = 1$. Which function is continuous on $(-\infty, +\infty)$?



Graph of f with only discontinuity at x=1

Нужно проверить: "Exact branch heights should be checked against the source image."

1. $|f(x) - 1|$.
2. $|f(x-1) - 1|$.
3. $|f(|x-1|) - 1|$.
4. $f(x - |x|)$.

QUESTION 26

Find

$$\lim_{\theta \to 0} \frac{1 - \cos\theta}{\sin\theta \arctan(2\theta)}.$$

1. $\frac{1}{2}$.
2. 1.
3. 0.
4. $\frac{1}{4}$.

QUESTION 27

Which graph has exactly one horizontal asymptote and no vertical asymptotes?

1. $y = \frac{1}{1+x^{2020}}$.
2. $y = \frac{1}{1+x^{2019}}$.
3. $y = \frac{1}{2020x-1}$.
4. $y = \frac{1}{2019x+1}$.

QUESTION 28

Find

$$\lim_{x \to 2} \frac{\ln(x-1)}{2x^2-3x-2}.$$

1. 0.
2. $\frac{1}{5}$.
3. ∞.
4. $\frac{1}{2}$.