Математический анализ 1 — МИЭФ, 2021 midterm

МИЭФ | Математический анализ 1 | 2021 | midterm

Рисунки пока рендерятся в тестовом режиме и могут отличаться от исходных материалов.

QUESTION 1

Consider

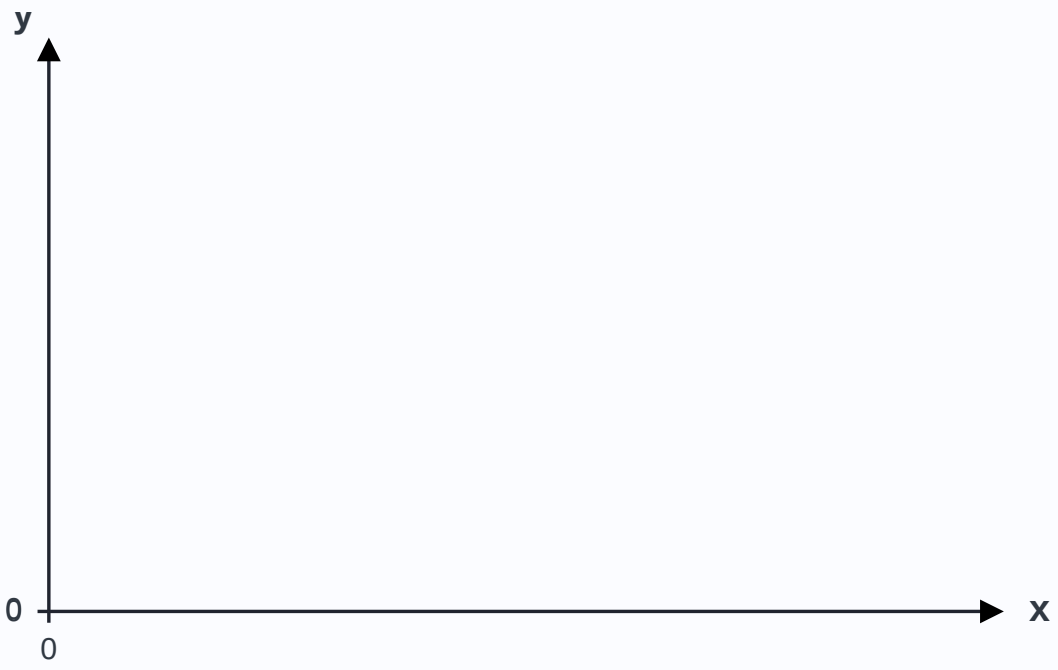
$$f(x) = \frac{|x-1|}{1-x^2}.$$

(a) Find the domain of f.

(b) Find and classify all discontinuities of f. Give equations of all vertical asymptotes, if any.

(c) Find the range of f. Justify.

(d) Sketch $y = |2f(-x)|$ and label the discontinuities.



Transforming f to |2f(-x)|

QUESTION 2

Let $f(x) = P(x)/Q(x)$, where P and Q are quadratic polynomials.

(a) Find a possible $Q(x)$ if f has a removable discontinuity at $x = 1$ and vertical asymptote $x = -2$.

(b) Using (a), find $f(x)$ if $f(2) = 0$ and $\lim_{x\to 1} f(x) = -3$. Explain every step.

(c) Using (b), find $\lim_{x\to -\infty} f(x)$, $\lim_{x\to 0} f(x)$, and $\lim_{x\to +\infty} f(x)$.

QUESTION 3

Consider

$$f(x) = \begin{cases} \left(\frac{x-1}{x}\right)^{-x}, & x < 0, \\ \sqrt{Ax^2 + Bx + C}, & x \geq 0. \end{cases}$$

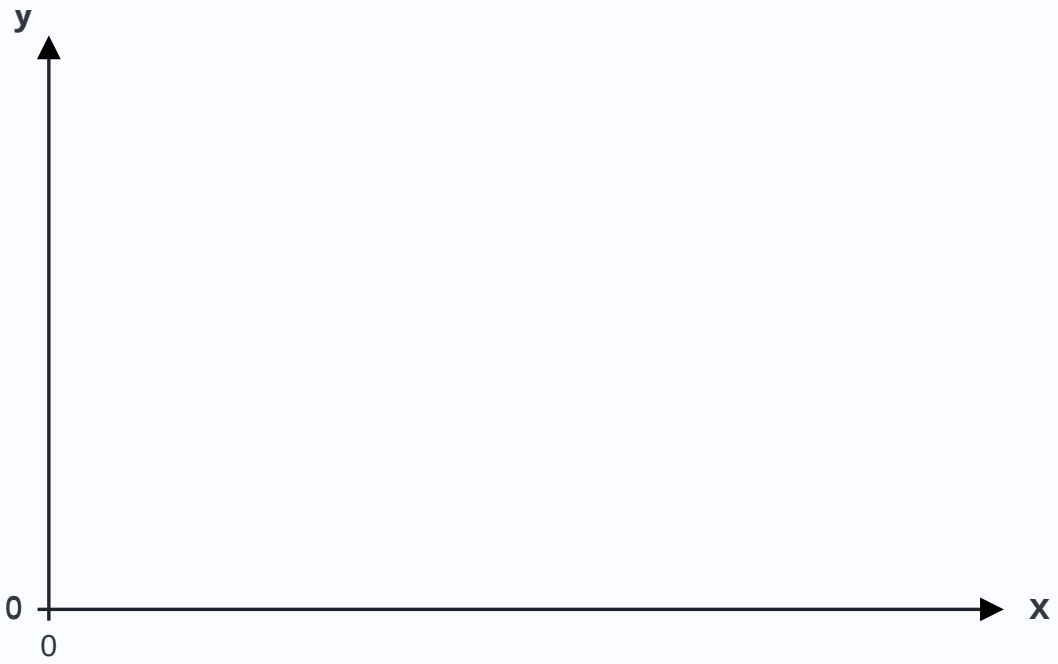
(a) Find $\lim_{x\to 0-} f(x)$.

(b) Find an asymptote of $y = f(x)$ as $x \to -\infty$, or explain why none exists.

(c) Find A, B, and C if f is continuous at 0 and has slant asymptote $y = x/2 - 2/3$ as $x \to +\infty$.

QUESTION 4

Sierpiński's construction starts with a 1×1 square. At each step, every remaining square is divided into 9 smaller squares and the middle square is removed.



Sierpiński carpet construction

(a) Let a_n be the area of an individual square after step n. Find a_1, a_2, a_3 and a_n.

(b) Let s_n be the number of squares after step n. Given $s_1 = 1$, $s_2 = 8$, find s_3, s_4 and s_n.

(c) Let c_n be the total area after step n. Find c_n and $\lim c_n$.

(d) How does the limiting area change if at each step the inscribed circles in the middle squares are removed instead of the whole middle squares?

QUESTION 5

Function f is defined for all $x \in \mathbb{R}$. It is continuous everywhere except at $x = -1$ and $x = 1$. The figure shows a portion of the graph $y = f(x)$.

Which of the following limits does not exist?

Graph of f with discontinuities at x=-1 and x=1

Нужно проверить: "Coordinates are approximate from the source graph."

1. $\lim_{x \to \pi} f(\cos x)$.
2. $\lim_{x \to \pi/6} f(2 \sin x)$.
3. $\lim_{x \to \pi/2} f(2 \cos x)$.
4. $\lim_{x \to \pi/3} f(\sin x)$.

QUESTION 6

Find

$$\lim_{x\to 0} \frac{\ln(\cos 2x)}{\arctan(\sqrt[3]{1-x^2}-1)}.$$

1. $\frac{3}{2}$.
2. -2.
3. $-\frac{2}{3}$.
4. 6.

QUESTION 7

Find the asymptote as $x \to -\infty$ for

$$g(x) = \frac{2x^2+5x+e^x}{\sqrt{4x^2+7}}.$$

1. $2y+2x=-5$.
2. $2y-2x=7$.
3. $2y-2x=5$.
4. $2y+2x=-7$.

QUESTION 8

Which expression is not an indeterminate form?

1. $\infty - \infty$.
2. 1^{∞}.
3. $\frac{\infty}{\infty}$.
4. $\frac{0}{\infty}$.

QUESTION 9

The terms of a decreasing sequence $\{a_n\}$ lie inside the interval

$$(10^{-24}, 1)$$

for all $n > 2020$.

Which statements must be true?

I. $\{a_n\}$ is bounded.

II. $\{a_n\}$ is convergent.

III. $\{a_n\}$ is infinitesimally small.

1. I and II only.
2. I only.
3. II and III only.
4. I, II and III.

QUESTION 10

Which statement is true about

$$\phi(x) = \frac{1}{\sqrt{x^4 - 8x}}?$$

1. The range of ϕ is $(0, +\infty)$.
2. The domain of ϕ is $(-\infty, 0] \cup [2, +\infty)$.
3. The graph $y = \phi(x)$ has two vertical asymptotes $x = 0$ and $x = -2$.
4. Function ϕ is decreasing.

QUESTION 11

Find $a + b$ given that

$$f(x) = \begin{cases} \log_3\left(\frac{2-x}{1-x}\right), & |x| < 1, \\ x^2 + ax + b, & |x| \geq 1 \end{cases}$$

is continuous.

1. $a + b = -\frac{1}{2}$.
2. $a + b = -1 + \frac{1}{\ln 3}$.
3. $a + b = 1$.
4. $a + b = \frac{\ln 3}{2}$.

QUESTION 12

Compute

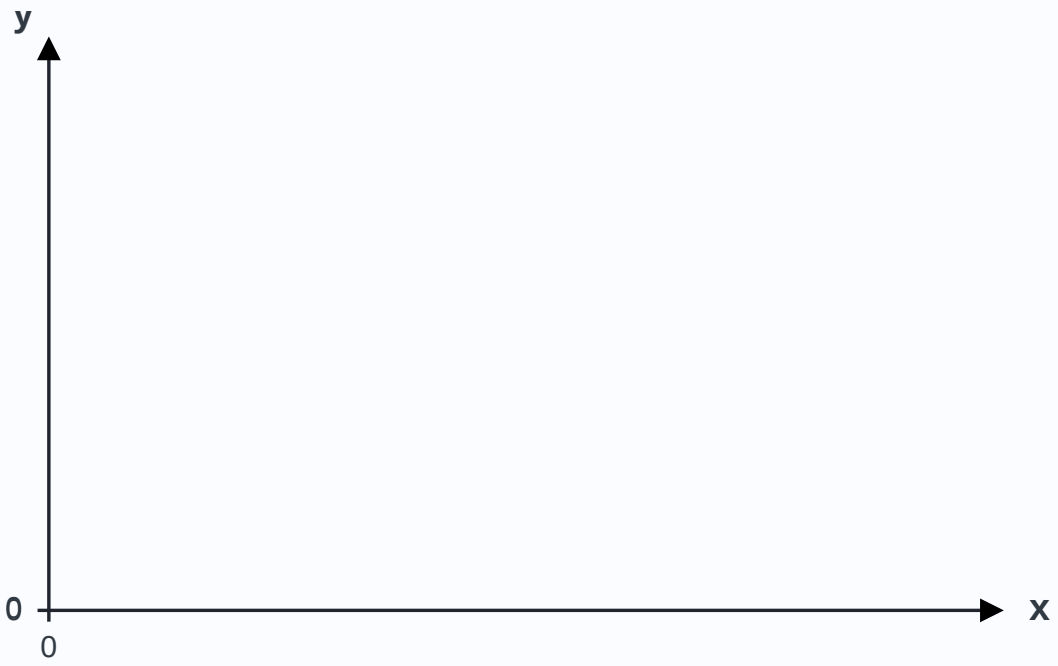
$$\lim_{n\to\infty} \frac{n^2}{n+(n+1)+(n+2)+\cdots+(2n-1)+2n}.$$

1. $\frac{1}{2}$.
2. 0.
3. ∞.
4. $\frac{2}{3}$.

QUESTION 13

Which graph could represent

$$y = f(|x-1|-1)-1, \qquad f(x) = x^2?$$



Four candidate graphs for y=(|x-1|-1)^2-1

Нужно проверить: "The source only gives small qualitative panels."

1. a.
2. b.
3. c.
4. d.

QUESTION 14

Find k such that

$$\ln\left(\frac{2^x+3^x-5^x}{x}\right) \sim kx \qquad \text{as } x \to 0.$$

1. 1.
2. $\ln\left(\frac{6}{5}\right)$.
3. 0.
4. No such k exists.

QUESTION 15

Find

$$\lim_{x\to 2}\left(\frac{x^2-3x+4}{x^2+2x-6}\right)^{2/(x-2)}.$$

1. e^{-10}.
2. e^{-5}.
3. $e^{-4/5}$.
4. e^{2}.

QUESTION 16

Function $f(x)$ is continuous for all real numbers. Which statements are true?

I. $\lim_{x\to 2020} f(x) = f(2020)$.

II. $\lim_{n\to\infty} f(x_n) = f(L)$ for every infinite sequence $\{x_n\}$ such that $\lim_{n\to\infty} x_n = L$.

III. If $f(a) > 0$, then there exists $\delta > 0$ such that $f(x) > 0$ for $x \in (a-\delta, a+\delta)$.

1. I, II and III.
2. I and III only.
3. II and III only.
4. I and II only.

QUESTION 17

The terms of infinite sequences $\{x_n\}$ and $\{y_n\}$ satisfy $0 < x_n < y_n$ for all n.

Which statements are true?

I. If $\{x_n\}$ and $\{y_n\}$ are both convergent, then $\left\{\frac{x_n}{y_n}\right\}$ is also convergent.

II. If $\{y_n\}$ is infinitesimally small, then $\{x_n\}$ is also infinitesimally small.

III. If $\{x_n\}$ is infinitely large, then $\{y_n\}$ is divergent.

1. I only.
2. I and II only.
3. II and III only.
4. I and III only.

QUESTION 18

Which functions have a jump discontinuity?

$$u(x) = \frac{x^2 - 1}{|x - 1|}, \qquad v(x) = \cos\left(\frac{x^2 - 1}{|x - 1|}\right), \qquad w(x) = \ln\left(2 + \frac{x^2 - 1}{|x - 1|}\right).$$

1. u only.
2. u and w.
3. u, v, and w.
4. None of them.

QUESTION 19

The sign function is

$$\operatorname{sgn} x = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0. \end{cases}$$

Which limit does not exist?

1. $\lim_{x \to 0+} \operatorname{sgn}(x)$.
2. $\lim_{x \to 1-} \operatorname{sgn}(x - 1)$.
3. $\lim_{x \to 2} \operatorname{sgn}(x - 2)$.
4. $\lim_{x \to 3} \operatorname{sgn}^2(x - 3)$.

QUESTION 20

Which infinite sequences are convergent?

I. $\left\{\frac{\sqrt[n]{n}}{2.4^n}\right\}$.

II. $\left\{\frac{\sqrt[n]{n}}{\sqrt[n]{n^3+2020}}\right\}$.

III. $\left\{\frac{\sqrt[n]{n}}{\ln(n^n)}\right\}$.

1. All three.
2. I and II only.
3. II and III only.
4. I and III only.

QUESTION 21

An even function $f(x)$ has a Type II discontinuity at $x = 0$. Which statements could be true?

I. $\frac{1}{f(x)}$ has a removable discontinuity at $x = 0$.

II. $\frac{1}{f(x)}$ has a jump discontinuity at $x = 0$.

III. $\frac{1}{f(x)}$ has a Type II discontinuity at $x = 0$.

1. I, II and III.
2. I and II only.
3. II and III only.
4. I and III only.

QUESTION 22

Find

$$\lim_{x \to -5} \frac{6 - \sqrt{41 + x}}{x + 5}.$$

1. -1.
2. $-\frac{1}{6}$.
3. $-\frac{1}{12}$.
4. The limit does not exist.

QUESTION 23

Find

$$\lim_{x \to +\infty} \frac{\sqrt{x} + 2\sqrt[3]{x} + 5\sqrt[6]{x}}{\sqrt{4x} + \sqrt[3]{1 + 3x}}.$$

1. $\frac{1}{2}$.
2. $\frac{1}{3}$.
3. $\frac{2}{3}$.
4. 1.

QUESTION 24

Let f be a strictly increasing continuous function defined for all x.

Which statements must be true?

I. $f^2(x)$ is increasing.

II. $f^3(x)$ is increasing.

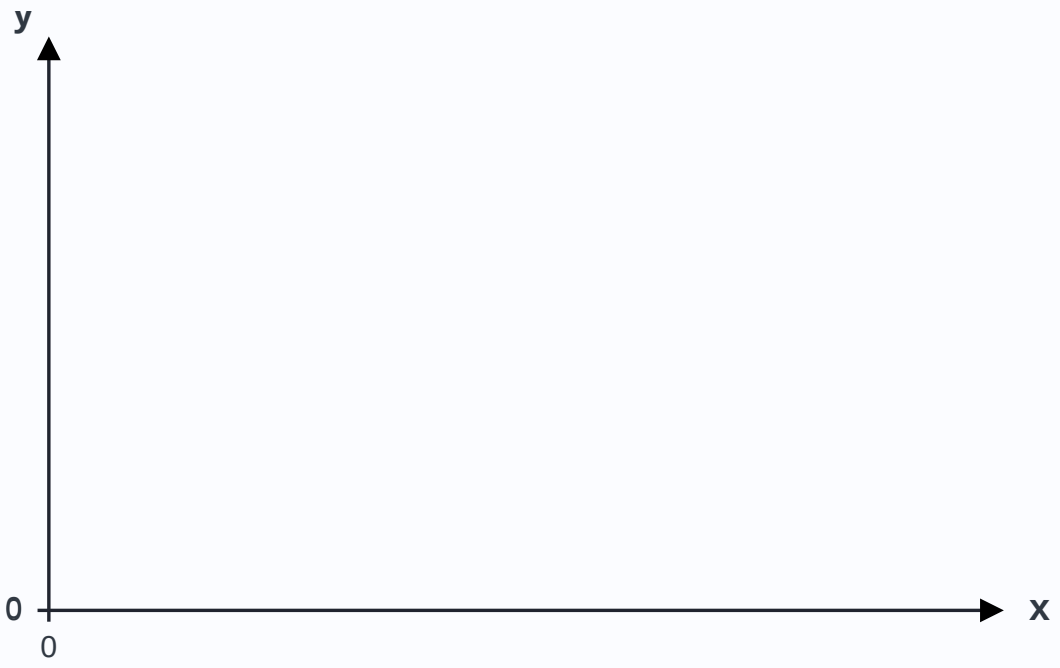
III. The inverse function $f^{-1}(x)$ is defined and increasing on the range of f.

1. I, II and III.
2. I and II only.
3. II and III only.
4. I and III only.

QUESTION 25

Function f is defined for all $x \in \mathbb{R}$ and is discontinuous at $x = -1$ and $x = 1$. The figure shows a portion of $y = f(x)$.

Which expression is the biggest?



Graph of f with discontinuities at -1 and 1

Нужно проверить: "Coordinates are approximate from the printed graph."

1. $\lim_{x \to -1+} f(x) - f(-1)$.
2. $f(-1) - \lim_{x \to -1-} f(x)$.
3. $f(1) - \lim_{x \to 1-} f(x)$.
4. $\lim_{x \to 1+} f(x) - f(1)$.

QUESTION 26

Find

$$\lim_{x \to +\infty} (3^x + 3^{2x})^{1/x}.$$

1. $3e^3$.
2. 9.
3. 12.
4. $3e$.

QUESTION 27

Which sequence is bounded?

1. $\sqrt{n}\cos\left(\frac{\pi n}{4}\right)$.
2. $1 - n + 2n - 3n^3$.
3. $2^{1-n+2n-3n^3}$.
4. $n^2 \sin\left(\frac{1}{n}\right)$.

QUESTION 28

Find

$$\lim_{x \to 2} \left(\frac{x^2 - 2x - 3}{2x^2 - 3x + 1} \right)^{x^2+1}.$$

1. 0.
2. -1.
3. ∞.
4. $\frac{1}{e}$.
5. The limit is not defined.