

Математический анализ 1 — МИЭФ, 2023 midterm

МИЭФ

Математический анализ 1

2023

midterm

Рисунки пока рендерятся в тестовом режиме и могут отличаться от исходных материалов.

QUESTION 1

Let $f(x) = e^{\sqrt{2x^2-1}}$.

- (a) Is f even, odd, or neither? Justify.
- (b) Find the domain of f . Justify.
- (c) Find the range of f . Justify.
- (d) Find all points where f is continuous. Justify.
- (e) Find $\lim_{h \rightarrow 0} \frac{\ln(f(1+h)) - \ln(f(1))}{h}$.

QUESTION 2

Let $f(x) = \frac{2x-2}{x^2+x-2}$.

- (a) For what x is f discontinuous?
- (b) At each discontinuity a , find $\lim_{x \rightarrow a} f(x)$ or justify non-existence.
- (c) Find all slant or vertical asymptotes. Justify.
- (d) A rational function $g(x) = \frac{b}{c+x}$ satisfies $f(x) = g(x)$ for all $x \in D[f]$. Find b, c .

QUESTION 3

Consider $a_n = \frac{2^n}{n!}$.

(a) Classify $\{a_n\}$ as increasing, nondecreasing, nonincreasing, decreasing, or neither. Justify.

(b) Is $\{a_n\}$ bounded? Justify.

(c) What conclusion follows from (a) and (b)? Name all theorems used.

(d) If $x_n \rightarrow L$, find $R = \lim_{n \rightarrow \infty} \frac{x_{n+1}}{x_n}$.

(e) Calculate $A = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$ for $a_n = 2^n/n!$. Are A and R consistent?

QUESTION 4

Let f be bounded and continuous on $[0, \infty)$ and $f(x) \neq 0$ for all $x \geq 0$. Which statements must be true?

(a) There exists $c \in [0, \infty)$ such that $f(x) \leq f(c)$ for all $x \geq 0$.

(b) f does not change sign on $[0, \infty)$.

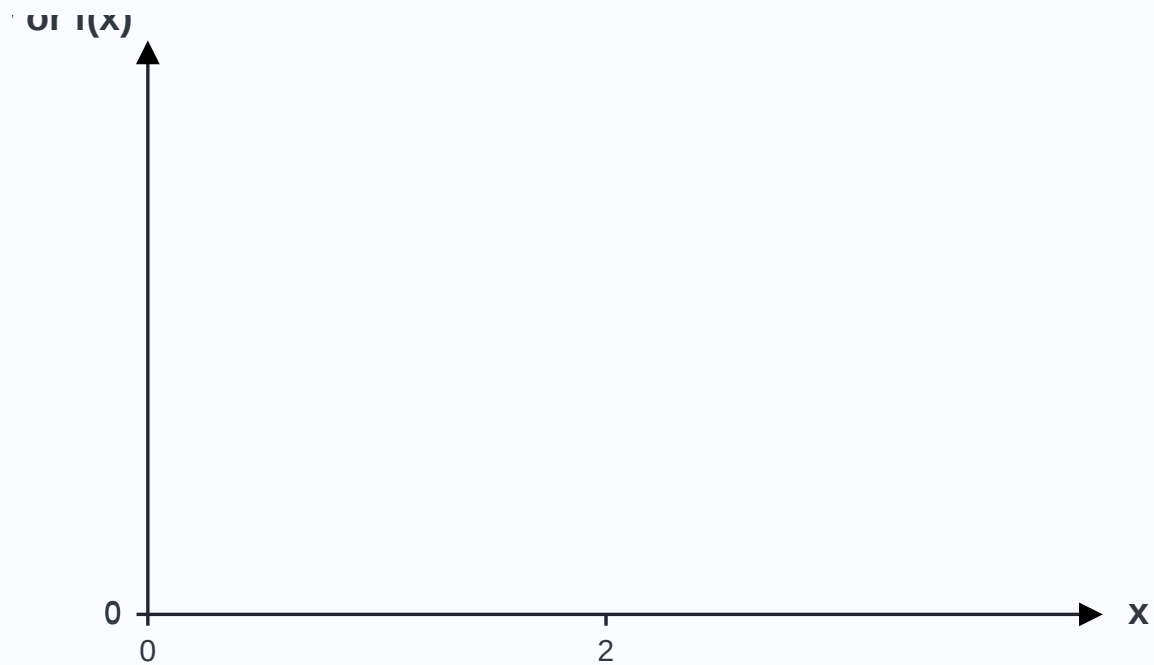
(c) $\lim_{x \rightarrow +\infty} f(x) = L$ for some finite L .

If a statement is always true, explain why and mention relevant theorems. If false, provide a counterexample.

QUESTION 5

The graph of $y = f(x)$ is shown below. Which of the alternatives **a–e** represents the graph of

$$y = f(|x|)?$$



Source graph of f and five candidate graphs for $f(|x|)$

Нужно проверить: "Coordinates are approximate because the source graph provides only tick marks, not an explicit formula."

1. a
2. b
3. c
4. d
5. e

QUESTION 6

Let

$$f(x) = 2x, \quad g(x) = \frac{4}{x-1}.$$

The complete set of solutions of

$$f(g(x)) = g(f(x))$$

is:

1. $x_1 = \frac{1}{3}$.
2. $x_1 = 2$.
3. $x_1 = 3$.
4. $x_1 = -1, x_2 = 2$.
5. $x_1 = \frac{1}{3}, x_2 = 2$.

QUESTION 7

Let

$$f(x) = x^5 - 1.$$

Then the inverse function f^{-1} is:

1.
$$f^{-1}(x) = \frac{1}{\sqrt[5]{x} + 1}.$$

2.
$$f^{-1}(x) = \frac{1}{\sqrt[5]{x+1}}.$$

3.
$$f^{-1}(x) = \sqrt[5]{x-1}.$$

4.
$$f^{-1}(x) = \sqrt[5]{x} - 1.$$

5.
$$f^{-1}(x) = \sqrt[5]{x+1}.$$

QUESTION 8

Find the principal period of

$$f(x) = 3 - 2 \cos^2 \left(\frac{\pi x}{3} \right).$$

1. 1.
2. 2.
3. 3.
4. 5.
5. 6.

QUESTION 9

Let

$$f(x) = \cos(\arctan x).$$

What is the range of f ?

1. $(0, 1)$.
2. $(0, 1]$.
3. $[0, 1]$.
4. $(-1, 1)$.
5. $[-1, 1]$.

QUESTION 10

Let

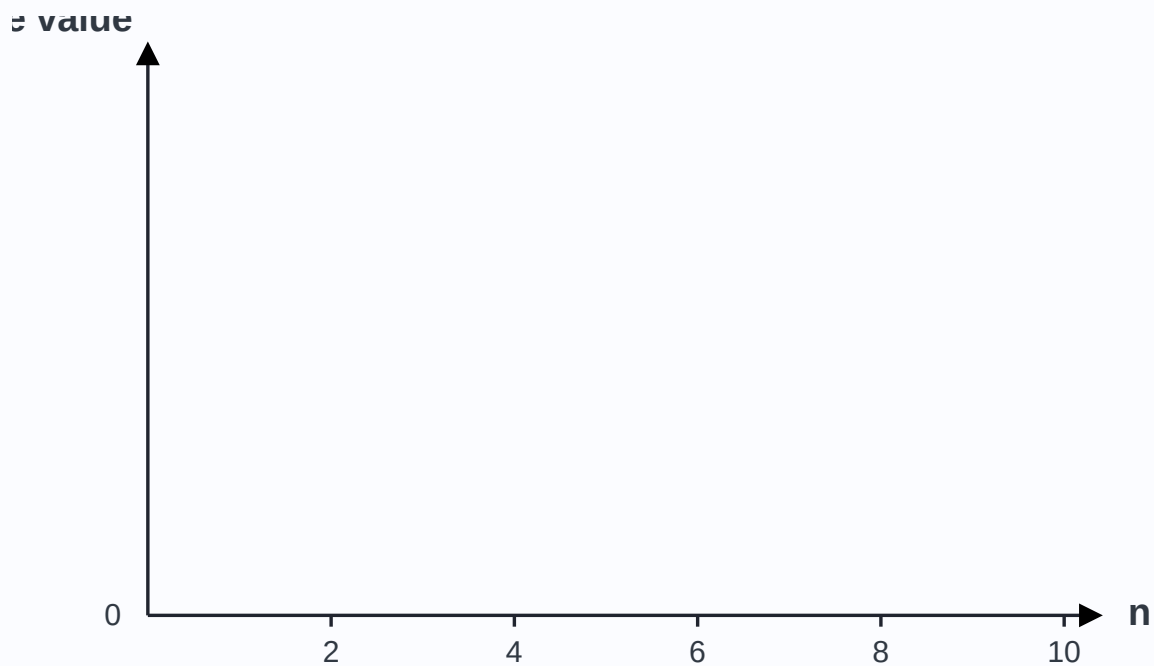
$$f(x) = \left| \sin x - \frac{1}{2} \right|.$$

The maximum value attained by f is:

1. at most $\frac{1}{2}$;
2. 1;
3. $\frac{3}{2}$;
4. $\frac{3}{4}$;
5. greater than $\frac{3}{2}$.

QUESTION 11

Match the five graphs with the five sequences.



Five discrete sequence graphs

Нужно проверить: "The plotted values are qualitative; exact numerical coordinates are not supplied."

Choose the correct matching:

1. $v_n = \frac{2a}{n+1}, \quad w_n = \frac{an}{n+1}, \quad x_n = \frac{a}{2^{n-1}}, \quad y_n = \frac{(a-2)^n}{n!}, \quad z_n = \frac{a-2}{\sqrt[n]{n}}.$

2. $v_n = \frac{a}{2^{n-1}}, \quad w_n = \frac{(a-2)^n}{n!}, \quad x_n = \frac{2a}{n+1}, \quad y_n = \frac{an}{n+1}, \quad z_n = \frac{a-2}{\sqrt[n]{n}}.$

3. $v_n = \frac{2a}{n+1}, \quad w_n = \frac{an}{n+1}, \quad x_n = \frac{a-2}{\sqrt[n]{n}}, \quad y_n = \frac{(a-2)^n}{n!}, \quad z_n = \frac{a}{2^{n-1}}.$

4. $v_n = \frac{(a-2)^n}{n!}, \quad w_n = \frac{a-2}{\sqrt[n]{n}}, \quad x_n = \frac{a}{2^{n-1}}, \quad y_n = \frac{2a}{n+1}, \quad z_n = \frac{an}{n+1}.$

5. $v_n = \frac{2a}{n+1}, \quad w_n = \frac{an}{n+1}, \quad x_n = \frac{(a-2)^n}{n!}, \quad y_n = \frac{a}{2^{n-1}}, \quad z_n = \frac{a-2}{\sqrt[n]{n}}.$

QUESTION 12

If

$$a_1 = 2, \quad a_n = a_{n-1} + \frac{1}{3}$$

for every integer $n > 1$, then a_{101} equals:

1. $35 - \frac{2}{3}.$

2. $35 - \frac{1}{3}.$

3. 35.

4. $35 + \frac{1}{3}.$

5. $35 + \frac{2}{3}.$

QUESTION 13

Which of the following sequences converge?

$$\text{I. } a_n = \frac{5n}{2n-1}, \quad \text{II. } c_n = \frac{e^n}{1+e^n}, \quad \text{III. } b_n = \frac{e^n}{n}.$$

1. I only.
2. II only.
3. I and II only.
4. I and III only.
5. I, II, and III.

QUESTION 14

Which statements are true?

- I. All bounded sequences are convergent.
- II. If a sequence is unbounded, then it is infinitely large.
- III. An unbounded sequence may have two limits.

1. I only.
2. II only.
3. II and III only.
4. I and III only.
5. None of the statements is true.

QUESTION 15

Find

$$\lim_{n \rightarrow \infty} \left(\frac{n^4 + 2n^2 + 4}{n^4 - 2n^2 + 4} \right)^{n^2}.$$

1. 1.
2. e .
3. e^2 .
4. e^4 .
5. $+\infty$.

QUESTION 16

Let sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be such that both difference sequences

$$\{x_n - y_n\} \quad \text{and} \quad \{y_n - z_n\}$$

are infinitesimally small.

Which of the following sequences must be convergent?

$$\text{I. } \{x_n - z_n\}, \quad \text{II. } \{x_n + z_n\}, \quad \text{III. } \left\{ \frac{x_n z_n}{y_n^2} \right\}.$$

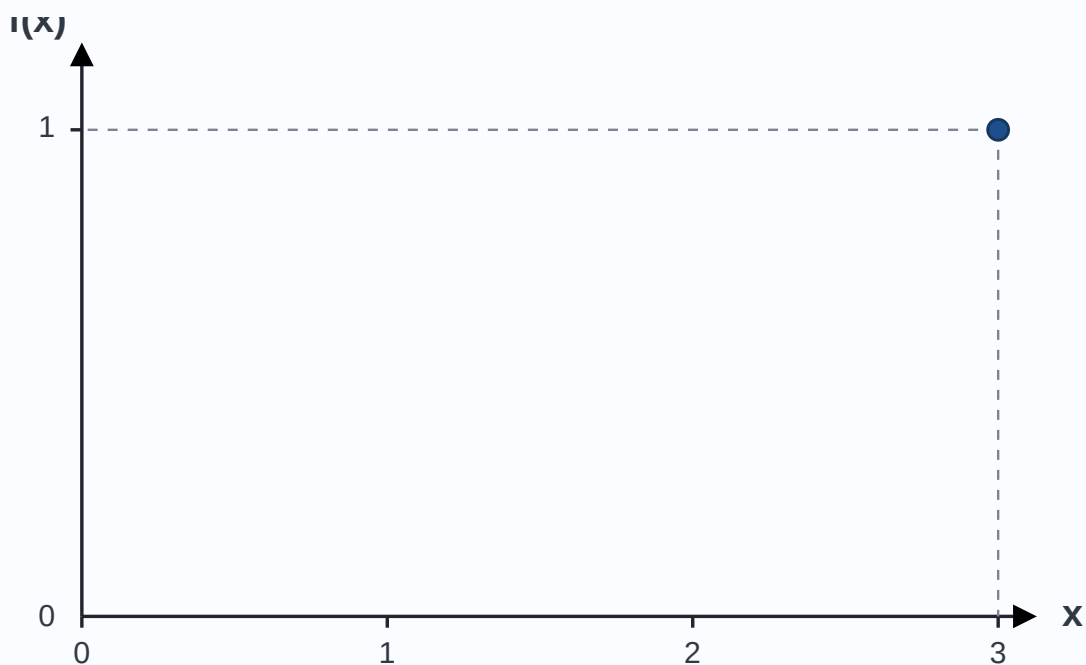
1. I only.
2. II only.
3. I and II only.
4. I and III only.
5. I, II, and III.

QUESTION 17

The graph of a function f is shown below.

For which values of c does

$$\lim_{x \rightarrow c} f(x) = 1?$$



Piecewise graph used to evaluate two-sided limits

Нужно проверить: "The exact height of the horizontal ray for $x > 3$ is not labelled; only that it is below 1 matters."

1. 0 only.
2. 0 and 3 only.
3. -2 and 0 only.
4. -2 and 3 only.
5. -2, 0, and 3.

QUESTION 18

Find

$$\lim_{x \rightarrow 0} \frac{e^{2x} - 1}{\tan x}.$$

1. The limit does not exist.
2. -1 .
3. 0 .
4. 1 .
5. 2 .

QUESTION 19

Among the following choices of δ , which is the largest that can be used successfully for arbitrary $\varepsilon > 0$ in an ε - δ proof of

$$\lim_{x \rightarrow 2} (1 - 3x) = -5?$$

1. $\delta = 3\varepsilon$.
2. $\delta = \varepsilon$.
3. $\delta = \frac{1}{2}\varepsilon$.
4. $\delta = \frac{1}{4}\varepsilon$.
5. $\delta = \frac{1}{5}\varepsilon$.

QUESTION 20

Let

$$f(x) = \sin\left(\frac{x+1}{x^2}\right).$$

Which statements are true?

- I. The graph of f has a horizontal asymptote $y = 0$.
- II. The graph of f has a horizontal asymptote $y = 1$.
- III. The graph of f has a vertical asymptote at $x = 0$.

- 1. I only.
- 2. II only.
- 3. III only.
- 4. I and III only.
- 5. II and III only.

QUESTION 21

Find the asymptote of

$$y = \frac{x^3}{\sqrt{x^4 + x^2 + 1}}$$

as $x \rightarrow -\infty$.

1. $y = 1$.
2. $y = x$.
3. $y = -x$.
4. $y = x + 1$.
5. $y = -x + 1$.

QUESTION 22

Let

$$f(x) = \frac{x^3}{x^2 - 4} + |x|.$$

The graph of $y = f(x)$ has:

1. two vertical asymptotes, one horizontal asymptote, and one slant asymptote with nonzero slope;
2. two vertical asymptotes and two slant asymptotes with slopes of different signs;
3. one vertical asymptote and two slant asymptotes with positive slopes;
4. one vertical asymptote and three slant asymptotes with positive slopes;
5. three different asymptotes.

QUESTION 23

Find

$$\lim_{x \rightarrow +\infty} \frac{\ln(1 + \sqrt{x} + \sqrt[3]{x})}{\ln(1 + \sqrt[3]{x} + \sqrt[4]{x})}.$$

1. 1.

2. 0.

3. $\frac{2}{3}$.

4. $\frac{3}{4}$.

5. $\frac{3}{2}$.

QUESTION 24

Let f be continuous at $x = 2$, and

$$f(x) = \begin{cases} \frac{\sqrt{2x+5} - \sqrt{x+7}}{x-2}, & x \neq 2, \\ k, & x = 2. \end{cases}$$

Then:

1. $k = 0$.

2. $k = \frac{1}{6}$.

3. $k = \frac{1}{3}$.

4. $k = 1$.

5. $k = \frac{7}{5}$.

QUESTION 25

Let f be a continuous function defined by

$$f(x) = \sqrt{\tan^2 x - 1}.$$

Which interval could be the domain of f ?

1. $\left(\frac{3\pi}{4}, \pi\right)$.
2. $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.
3. $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$.
4. $\left(-\frac{\pi}{4}, 0\right)$.
5. $\left(-\frac{3\pi}{4}, -\frac{\pi}{4}\right)$.

QUESTION 26

Let f be continuous on the closed interval $[-3, 6]$. If

$$f(-3) = -1, \quad f(6) = 3,$$

then the Intermediate Value Theorem guarantees that:

1. $f(0) = 0$;
2. $f(x) \leq f(c)$ for some $c \in [-3, 6]$;
3. $-1 \leq f(x) \leq 3$ for every x between -3 and 6 ;
4. $f(c) = 0$ for at least one c between -1 and 3 ;
5. $f(c) = 1$ for at least one c between -3 and 6 .

QUESTION 27

According to the Intermediate Value Theorem, which statements are true?

I. The equation

$$x^{20} + x^{23} - 2022 = 0$$

has at least one root in $[-10, 10]$.

II. The function

$$f(x) = \log_2 x$$

takes the value 11 in $[2022, 2202]$.

III. The function

$$f(x) = \frac{1}{x^2 - 6x + 8}$$

takes the value 0 in $[1, 3]$.

1. III only.
2. I and III only.
3. II and III only.
4. I and II only.
5. I, II, and III.

QUESTION 28

Let $a, b, c, d \in \mathbb{R}$. The polynomial equation

$$x^7 + ax^5 + bx^3 + cx + d = 0$$

must have:

1. only one root;
2. at least one root;
3. an even number of roots;
4. no negative roots;
5. no positive roots.

QUESTION 29

At how many points do the graphs

$$y = x^{12} \quad \text{and} \quad y = 2^x$$

intersect?

1. None.
2. One.
3. Two.
4. Three.
5. Four.

QUESTION 30

Let k be the number of real solutions of

$$e^x + x - 2 = 0$$

in $[0, 1]$, and let n be the number of real solutions outside $[0, 1]$.

Which statement is true?

1. $k = 0$ and $n = 1$.
2. $k = 1$ and $n = 0$.
3. $k = n = 1$.
4. $k > 1$.
5. $n > 1$.

QUESTION 31

Let

$$f(x) = \begin{cases} \frac{\sqrt{x^2} - x}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Which statement is true?

1. f is continuous for every x .
2. f has a removable discontinuity at $x = 0$.
3. f has a jump discontinuity at $x = 0$, and

$$\lim_{x \rightarrow 0^+} f(x) = f(0).$$

4. f has a jump discontinuity at $x = 0$, and

$$\lim_{x \rightarrow 0^-} f(x) = f(0).$$

5. f has a Type II discontinuity at $x = 0$.

QUESTION 32

Consider

$$f(x) = \begin{cases} 2^{1/x}, & x < 0, \\ 1, & 0 \leq x \leq 1, \\ \ln(x^2 + x - 2), & x > 1. \end{cases}$$

The function f has:

1. only one jump discontinuity;
2. two jump discontinuities;
3. one Type II discontinuity only;
4. one jump discontinuity and one Type II discontinuity;
5. one jump discontinuity and one removable discontinuity.

QUESTION 33

Consider

$$f(x) = \arctan\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-2}\right).$$

Let D be the number of Type II discontinuities of f , and let A be the number of distinct asymptotes of f .

Find $D + A$.

1. 1.
2. 3.
3. 6.
4. 2.
5. 8.

QUESTION 34

A function f has a Type II discontinuity at $x = a$.

Which of the following statements may be true?

I. The left-hand limit

$$\lim_{x \rightarrow a^-} f(x)$$

exists.

II. $f(a)$ exists.

III. f is bounded in some neighbourhood of a .

1. II only.
2. III only.
3. I and II only.
4. II and III only.
5. I, II, and III.